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2007 J. Phys. A: Math. Theor. 40 6965

(http://iopscience.iop.org/1751-8121/40/25/S47)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 6965–6971 doi:10.1088/1751-8113/40/25/S47

Gauge symmetry and consistent spin-2 theories

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Received 29 October 2006, in final form 21 December 2006
Published 6 June 2007
Online at stacks.iop.org/JPhysA/40/6965

Abstract
We study Lagrangians with the minimal amount of gauge symmetry required to propagate spin-2 particles without ghosts or tachyons. In general, these Lagrangians also have a scalar mode in their spectrum. We find that, in two cases, the symmetry can be enhanced to a larger group: the whole group of diffeomorphisms or an enhancement involving a Weyl symmetry. We consider the nonlinear completions of these theories. The intuitive completions yield the usual scalar–tensor theories except for the pure spin-2 cases, which correspond to two inequivalent Lagrangians giving rise to Einstein's equations. A more constructive self-consistent approach yields a background-dependent Lagrangian.

PACS numbers: 04.60.−m, 11.15.−q

1. Introduction

It has long been known that the minimal amount of gauge symmetry required for building the Hilbert space of massless spin-2 particles from tensorial objects are the linear 'transverse' diffeomorphisms [1] which we will call TDiff (see also [2, 3] for previous related work) and which are given by the following transformation:

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}, \qquad \partial_\mu \xi^\mu = 0. \tag{1}$$

The basic reason for that is that the trace $h = \eta^{\mu\nu}h_{\mu\nu}$ is Lorentz invariant and thus we can restrict to gauge symmetries which leave it invariant and just act on the spin-1 part of the symmetric tensor [1, 4]. In the first part of this paper we will study the general Lagrangians meeting the requirement of TDiff gauge invariance.

Besides, it is usually claimed that from the consistent self-interaction of spin-2 particle, the whole group of diffeomorphisms (Diff) is obtained [5–8]. These approaches are based on the imposition of the whole Diff at a linear level. It is then interesting to study how these results are modified for the TDiff case. We will be concerned with this in the second part of this paper.

2. Lorentz invariant healthy Lagrangians

We will first study the ghost and tachyon free Lagrangians for a symmetric tensor $h_{\mu\nu}$ that incorporate the TDiff gauge symmetry. The most general second-order Lagrangian for this field is

$$\mathcal{L} = \frac{1}{4}\partial_\mu h^{\nu\rho}\partial^\mu h_{\nu\rho} - \frac{\beta}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu_\rho + \frac{a}{2}\partial^\mu h \partial^\rho h_{\mu\rho} - \frac{b}{4}\partial_\mu h \partial^\mu h - \frac{1}{2}m^2(h^2 - \alpha h_{\mu\nu}h^{\mu\nu}), \quad (2)$$

where the first term is mandatory to propagate spin-2 polarizations and indices have been manipulated with the Minkowski metric $\eta_{\mu\nu}$. Let us consider a general gauge transformation

$$\delta h_{\mu\nu} = \tau_{\mu\nu} + \frac{1}{n}\phi\eta_{\mu\nu} \quad (3)$$

where $\tau^\mu{}_\mu = 0$ and n is the dimension of the spacetime. One can prove (cf [4]) that this Lagrangian can have a gauge symmetry only for the case $\beta = 1$ and

$$\tau_{\mu\nu} = \partial_{(\mu}\zeta_{\nu)} - \frac{1}{n}\partial_\alpha\zeta^\alpha\eta_{\mu\nu}. \quad (4)$$

After imposing $\beta = 1$, the different choices of parameters in (2) which give rise to a gauge symmetry, together with the conditions on the parameters of the gauge transformation (3) are as follows.

- TDiff: $\alpha = 0$, $\phi = 0, \partial_\rho\zeta^\rho = 0$.
- Weyl: $\alpha = n, a = \frac{2}{n}, b = \frac{n+2}{n^2}$, $\zeta_\rho = 0$.
- Diff: $m^2 = 0, a = b = 1$, $\phi = \partial_\rho\zeta^\rho$.
- Weyl and TDiff (WTDiff): $m^2 = 0, a = \frac{2}{n}, b = \frac{n+2}{n^2}$, $\partial_\rho\zeta^\rho = 0$.

The previous parameters are unique up to for field redefinition $h_{\mu\nu} \mapsto h_{\mu\nu} + \lambda h\eta_{\mu\nu}$. We see that the TDiff invariance requirement implies $\beta = 1$ and $\alpha = 0$. The first of these conditions is necessary also from the direct analysis of the propagating fields in the theory. To show this, we first decompose the tensor $h_{\mu\nu}$ into irreducible representations under the $SO(3)$ subgroup of the Lorentz group,

$$h_{00} = A, \qquad h_{0i} = \partial_i B + V_i, \qquad h_{ij} = \psi\delta_{ij} + \partial_i\partial_j E + 2\partial_{(i}F_{j)} + t_{ij}, \quad (5)$$

where $\partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t^i{}_i = 0$. At the linear level the different representations decouple, and thus we can study each of them independently. Studying the vector degrees of freedom (V_i, F_i) we realize that their Lagrangian can be expressed as

$$\mathcal{L}_v = -\frac{\Delta}{2}(V_i - \partial_0 F_i)^2 + \frac{1}{2}(\beta - 1)(\partial_0 V_i - \Delta F_i)^2, \quad (6)$$

where $\Delta = \partial_i\partial_i$. This Lagrangian has a ghost unless $\beta = 1$ [4]. We restrict our study to this kind of Lagrangians which, as we saw before, are TDiff gauge invariant.

Let us first study the massless case $m^2 = 0$. The Lagrangian for the tensor modes t_{ij} is simply

$$\mathcal{L}_t = -\tfrac{1}{4}t^{ij}\Box t_{ij}. \quad (7)$$

The field V_i is not dynamical and it gives rise to a constraint which also cancels F_i, making the vector sector trivial. The scalar sector (A, B, ψ, E) is more interesting. The field B is a Lagrange multiplier whose variation produces the constraint

$$(n-2)\psi = (a-1)h, \quad (8)$$

which once substituted back in the Lagrangian gives rise to

$$\mathcal{L}_s = \frac{C}{4} h \Box h \tag{9}$$

where $C = b - \frac{1-2a+(n-1)a^2}{n-2}$. Thus, the extra degree of freedom in the theory cancels whenever $C = 0$ and it is well behaved for $C < 0$. This means, that from the ghost and tachyon free condition we do not *only* recover the massless Fierz–Pauli Lagrangian, but rather a perfectly well-defined family of Lagrangians which propagate a spin-2 particle and a scalar. If we want to restrict ourselves to pure spin-2 we find two possibilities which coincide with the choices of enhanced gauge symmetries (see the previous page).

- $a = b = 1$ and field redefinitions $h_{\mu\nu} \mapsto h_{\mu\nu} + \lambda h \eta$, leading to the whole Diff group [9].
- $a = \frac{2}{n}$, $b = \frac{n+2}{n^2}$ which lead to the Weyl and TDiff gauge symmetry (WTDiff)[1].

For the massive case, a similar analysis yields the Fierz–Pauli massive Lagrangian $\beta = a = b = \alpha = 1$ as the only possibility.

3. Nonlinear completions

From the *strong equivalence principle*, gravity must couple to any kind of energy including its own [10]. Thus, if the graviton is described by a spin-2 particle, this particle must be coupled to its own energy–momentum tensor. If we do this at linear level, namely if we write the energy–momentum tensor of the graviton as the source for its equations of motion, this system of equations is no longer derivable from a Lagrangian and we need to write more nonlinear terms. This process goes on (see [11] and references therein) and it is usually stated that the only solution to this nonlinear series is general relativity with the usual Einstein–Hilbert action.

However, as we highlighted in the introduction, most of these approaches depart from the massless Fierz–Pauli Lagrangian (see however [1]). As we saw in the previous section, we could consider any of the well-behaved Lagrangians at the linear level as our starting point and try to find its nonlinear completion. We can do it intuitively [1, 4] and also more constructively (see below).

3.1. Intuitive completion

A possible nonlinear extension of the linear TDiff is provided by any subgroup of the nonlinear Diff for which an object f, which at the linear level reduces to the trace h, transforms as a scalar[2]. That is, given

$$f(\eta_{\mu\nu}, g_{\mu\nu}) = k + \eta^{\mu\nu} h_{\mu\nu} + O\left(h_{\mu\nu}^2\right) \tag{10}$$

for k a constant and $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, we want to find the subgroup of Diff such that

$$\delta_\xi f = \xi^\mu \partial_\mu f, \tag{11}$$

for $\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$. Clearly this subgroup, if it exists, will be background dependent. The previous condition can be expressed as

$$A^\mu_\rho \nabla_\mu \xi^\rho - \xi^\rho \partial_\rho f = A^\mu_\rho \partial_\mu \xi^\rho = 0, \tag{12}$$

[1] It may seem that we recover this possibility for $\lambda = -\frac{1}{n}$ in the previous transformation, but note that in this case the transformation is singular.

[2] We restrict to this possibility even if more general transformations could arise. A constructive way of finding this transformations will be discussed later. Also we restrict to those objects f of the form (10) for the Minkowski metric, but another background could be chosen.

where

$$A_\rho^\mu = 2\frac{\delta f}{\delta g_{\mu\nu}} g_{\nu\rho}.$$

In particular this means that the translations are always a subgroup.

Let us study the group structure for a generic f. From Frobenius theorem applied to the Diff, the infinitesimal transformations will be integrable iff [8]

$$\left[\xi_1^\mu \partial_\mu, \xi_2^\nu \partial_\nu\right] = \xi_3^\nu \partial_\nu \tag{13}$$

with $\xi_3^\nu = \xi_1^\mu \partial_\mu \xi_2^\nu - \xi_2^\mu \partial_\mu \xi_1^\nu$. The integrability condition is just

$$A_\rho^\mu \partial_\mu \xi_3^\rho = 2A_\rho^\mu \left(\partial_\mu \xi_{[1|}^\alpha \partial_\alpha \xi_{|2]}^\rho + \xi_{[1|}^\alpha \partial_\mu \partial_\alpha \xi_{|2]}^\rho\right) = 0, \tag{14}$$

for ξ_1 and ξ_2 satisfying (12). For the term involving second derivatives to cancel, the only possibility is $A_\rho^\mu = l(x) S_\mu^\rho$, with S_ρ^μ being a constant matrix, i.e.

$$2\delta f = l(x) g^{\mu\nu} \delta g_{\mu\nu} = l(x) g^{-1} \delta g, \tag{15}$$

where $g = \det g_{\mu\nu}$. Thus, f depends just on the determinant of the metric. The subgroup which preserves these functions will be TDiff also at the nonlinear level,

$$\partial_\mu \xi^\mu = 0. \tag{16}$$

Once integrated, this subgroup gives rise to the diffeomorphisms of Jacobian equal to 1, which are related to unimodular gravity [1].

Let us consider the simplest function $f = |g|$. We know that

$$|g| = 1 + \eta^{\mu\nu} h_{\mu\nu} + O\left(h_{\mu\nu}^2\right), \tag{17}$$

which in fact holds for any background. General Lagrangians where $|g|$ is considered as an independent degree of freedom have been studied in [1, 4] and they are usually equivalent to scalar–tensor theories of gravity except for an integration constant. It is interesting to note that once the Weyl symmetry

$$\delta g_{\mu\nu} = e^\phi g_{\mu\nu} \tag{18}$$

is also promoted as a gauge symmetry, we find a unique Lagrangian[3]

$$\mathcal{S}_{\mathrm{WTDiff}} = \int \mathrm{d}^4 x\, \hat{g}^{\mu\nu} R_{\mu\nu}(\hat{g}_{\mu\nu}) + S_M(g, \hat{g}_{\mu\nu}, \psi), \tag{19}$$

where $\hat{g}_{\mu\nu} = |g|^{-1/n} g_{\mu\nu}$ and S_M refers to a matter Lagrangian compatible with the Weyl symmetry. This Lagrangian yields Einstein's equations of motion in the gauge $|g| = 1$ (even when coupled to matter) except for the origin of the cosmological constant which comes from an integration constant [4].

Also note that (18) could be considered as too restrictive, as what we seek is a transformation of the determinant of the form

$$\delta_{(\phi,\xi)} g = \phi g + \xi^\mu \partial_\mu g. \tag{20}$$

However, from the previous expression we find that

$$[\delta_{(\phi_1,\xi_1)}, \delta_{(\phi_1,\xi_1)}] = \delta_{(\xi_{[1}\partial\phi_{2]},\xi_3)}. \tag{21}$$

If we want the same algebra to hold for the metric field $g_{\mu\nu}$ then it is clear that the transformation of the whole metric must be the usual conformal rescaling, i.e.

$$\delta_{(\phi,\xi)} g_{\mu\nu} = \phi^{1/n} g_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}. \tag{22}$$

[3] Note that this Lagrangian cannot be put in the Einstein frame, as it is Weyl invariant.

3.2. Constructive completion

There are different ways in which the nonlinear completion can be found constructively. The most direct one is to consider the energy–momentum tensor of the graviton as a source for the equations of motion of the graviton. This amounts to the first correction, or three-graviton vertex, for the linear action and is not consistent as there is no Lagrangian that gives rise to these equations of motion [5, 11]. Another way of performing the completion is to first show how the gauge symmetry can be enlarged nonlinearly [5, 8, 12] and then building a Lagrangian endowed with the nonlinear gauge symmetry up to the desired order. For the case of linearized Diff symmetry these nonlinear deformations were first addressed in [5] and later in [8, 12]. The equivalent calculation for TDiff and WTDiff is quite cumbersome and will be presented elsewhere [13].

An alternative approach for the Diff case which extends easily to the WTDiff case exists. This approach is based on the first-order formulation of gravity [6]. The second-order action for the first-order formulation of the Lagrangian (19) is

$$S^{(1)} = \int \delta^n x \{ -\hat{h}^{\mu\nu} 2\partial_{[\mu}\Gamma_{\rho]\nu}{}^{\rho} + \eta^{\mu\nu} 2\Gamma_{\lambda[\mu}{}^{\rho}\Gamma_{\rho]\nu}{}^{\lambda} \} \tag{23}$$

where $\hat{h}_{\mu\nu} = h_{\mu\nu} - h\eta_{\mu\nu}$ and the metric and the connection are now considered as independent fields. The equations of motion from the variation of $\hat{h}_{\mu\nu}$ are the traceless part of the Fierz–Pauli case, whereas from the variation of $\Gamma_{\mu\nu}{}^{\rho}$ we find a constraint for this field which, once solved, yields (for $n \neq 2$)

$$\Gamma_{\mu\nu}{}^{\rho} = \tfrac{1}{2}\eta^{\rho\sigma}(\partial_\mu \hat{h}_{\nu\sigma} + \partial_\nu \hat{h}_{\mu\sigma} - \partial_\sigma \hat{h}_{\mu\nu}). \tag{24}$$

This is just the equation of compatibility of the connection and the traceless metric at a linear order. Substituting this constraint in the Lagrangian we just get the WTDiff Lagrangian for $h_{\mu\nu}$. To calculate the energy–momentum tensor we use the Rosenfeld prescription for which we need to assign a weight to the fields $\hat{h}_{\mu\nu}$ and $\Gamma_{\mu\nu}{}^{\rho}$ which is the strongest assumption of Deser's method [11]. If we consider $\hat{h}^{\mu\nu}$ to be a tensor density and the indices of the connection to be vectorial, it is easy to see that the energy–momentum tensor is given by the usual energy–momentum tensor of [6] except for the fact that the tensor $\hat{h}_{\mu\nu}$ is now traceless. The WTDiff gauge symmetry implies that the object to couple to the free equations of motion of (23) is the traceless part of the energy–momentum tensor. Following [6], this coupling can be derived from the term

$$\mathcal{S}^{(2)} = -2 \int \mathrm{d}^n x \, \hat{h}^{\mu\nu} \Gamma_{\rho[\mu}{}^{\sigma} \Gamma_{\sigma]\nu}{}^{\rho}, \tag{25}$$

as $\hat{h}^{\mu\nu}$ is already traceless. Thus the action at third-order simply reads

$$\mathcal{S} \equiv \mathcal{S}^{(1)} + \mathcal{S}^{(2)} = \int \delta^n x \tilde{g}^{\mu\nu} R_{\mu\nu}(\Gamma_{\alpha\beta}{}^{\rho}), \tag{26}$$

where $\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \hat{h}^{\mu\nu}$. This Lagrangian differs from that which we guessed intuitively and is background dependent as $\hat{h}_{\mu\nu}$ involves $\eta_{\mu\nu}$ in its definition[4]. Besides, the equations of motion are not Einstein's equations but rather

$$R[\tilde{g}]_{\mu\nu} - \frac{1}{n}\eta_{\mu\nu} R[\tilde{g}] = 0 \tag{27}$$

where the connection is compatible with the metric associated with the density tensor $\tilde{g}^{\mu\nu}$, $g_{\mu\nu}$ which satisfies the constraint

$$\sqrt{-g} g^{\mu\nu}\eta_{\mu\nu} = n. \tag{28}$$

[4] Note that the nonlinear TDiff of the previous subsection depend only on a volume form.

This condition is preserved by diffeomorphisms satisfying

$$\eta_{\mu\nu}\left(g^{\mu\alpha}\delta^{\nu}_{\beta} - \tfrac{1}{2}\delta^{\alpha}_{\beta}g^{\mu\nu}\right)\nabla_{\alpha}\xi^{\beta} = 0, \tag{29}$$

which reduces to the transverse condition at the linear level. Again, the algebra of these diffeomorphisms does not close for general metrics and thus does not constitute a subgroup. Even vacuum solutions for equation (27) differ from Einstein's equations and we leave for future work the actual computation of phenomenological constraints of the theory which appear at the nonlinear level [13].

The reason why we have not found the WTDiff Lagrangian at the nonlinear level is the highly nonlinear dependence of the determinant of the metric as expressed in terms of traces. Remember that for the nonlinear WTDiff the action is expressed in terms of a tensor with $\hat{g} = 1$. It is impossible to find this condition for the determinant from the linear conditions on the trace with respect to the metric $\eta_{\mu\nu}$ in one single step. However, considering the field $\hat{h}_{\mu\nu}$ as a tensor density, the arguments of [6] also apply here and we are directly selecting condition (28) as the nonlinear one.

4. Conclusions

In this paper, we have shown that the requirement of ghost and tachyon free Lagrangians which describe spin-2 particles is satisfied by a whole family of Lagrangians at the linear level which satisfy a reducible gauge symmetry, TDiff. In all the cases but two there is a scalar mode which propagates whose mass is constrained by usual phenomenological bounds [4, 10]. The two special cases are the usual Diff case, where the theory becomes irreducible and WTDiff, which incorporates a Weyl gauge transformation.

We have seen that the intuitive nonlinear completion of these theories gives rise to Lagrangians which differ from the usual Einstein–Hilbert Lagrangian but which reduce again to scalar–tensor theories of gravity except for the previous two cases. In these cases, the equations of motion are Einstein's equations in a certain gauge except for the origin of the cosmological constant as happens in unimodular gravity [14, 15]. A more constructive approach to the nonlinear theory is possible in two ways. First, we can work in the first-order formalism of gravity and consistently couple the conserved energy–momentum tensor of the free action as a source of the equations of motion for the graviton itself. The standard derivation of [6] holds except for the fact that the metric must satisfy the extra constraint (28). This way we find a consistently-coupled Lagrangian (26) which does not produce Einstein's equations at the nonlinear level and is background dependent. This illustrates the non-uniqueness of the derivation in [6]. It is not clear whether other choices of linear variables or of weights for $\hat{h}^{\mu\nu}$ and $\Gamma^{\mu}{}_{\rho\sigma}$ exist that can reproduce the Lagrangian (19) using similar methods.

The other possibility is to deform the linear algebra and construct the nonlinear group of symmetry. This approach is currently under investigation [13].

Acknowledgments

I would like to thank E Álvarez, J Garriga and E Verdaguer for useful discussions and for a critical reading of this paper.

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